Computing Stability Criteria of Liquid Layers Spread over a Segment of a Sphere

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THE recent paper by Anliker and Beam[1] develops the stability of liquid layers spread over simple curved bodies and leads to an infinite set of linear algebraic equations as the stability criteria for the case of a sphere. The coefficients of the linear equations consist of integrals of the product of two associated Legendre functions of degrees which have to be determined from the boundary condition. It is the task of this paper to show recurrence relations among the coefficients to make it feasible to use a high-speed computer for a further study of the equation.

According to Ref. 1, the following equation should be set up, dropping the index m except for L and M:

$$\sigma \frac{F_j(a)}{F_j{}'(a)} L_{jj}{}^m h_j + \frac{T}{\rho a^2} [2 - p_j(p_j + 1)] L_{jj}{}^m h_j - g \sum_{n=1}^{\infty} M_{jn}{}^m h_n = 0$$

and determine if $\Delta(\sigma) = 0$, where $\Delta(\sigma)$ is the determinant of the coefficients, and h_n are the unknowns ($j = 1, 2, \ldots$; $m = 0, 1, 2, \ldots$).

To unify formulas, adopt the following definitions:

$$P_n(x) = (1/2^n n!)(d^n/dx^n)(x^2 - 1)^n$$
$$P_n{}^m(x) = (-1)^m (1 - x^2)^{m/2}(d^m/dx^m)P_n(x) \qquad m \geq 0$$

for the Legendre polynomials and associated Legendre functions, respectively. First find k such that, for a given m,

$$(d/d\theta)P_k{}^m(\cos\theta)|_{\theta=\alpha} = 0 \tag{1}$$

where α is one half of the angle of a segment of a sphere (see Ref. 1). From the formulas[2-4]

$$(d/d\theta)P_k{}^m(x) = (m + k)(m - k - 1)P_k{}^{m-1}(x) - m \cot\theta \, P_k{}^m(x) \tag{2}$$

$$(2k + 1) \sin\theta \, P_k{}^m(x) = P_{k-1}{}^{m+1}(x) - P_{k+1}{}^{m+1}(x) \tag{3}$$

$$(2k + 1)\, x \, P_k{}^m(x) = (k - m + 1)P_{k+1}{}^m(x) + (k + m)P_{k-1}{}^m(x) \tag{4}$$

where $x = \cos\theta$, Eq. (1) becomes

$$(k + 2)(m + k + 1)P_k{}^m(\cos\alpha) + (k + 1)(m - k - 2)P_{k+2}{}^m(\cos\alpha) = 0 \tag{5}$$

Now Eq. (5) can be solved for k by constructing $P_k{}^m(\cos\alpha)$ by a recurrence formula [e.g., Eq. (4)] for a given m. Denote such k's by $k_n (n = 1, 2, 3, \ldots)$.

To obtain $M_{jn}{}^m$ it can be seen that

$$(2k_j + 1)M_{jn}{}^m = (k_j - m + 1)G^m_{k_j+1,k_n} + (k_j + m)G^m_{k_j-1,k_n} \tag{6}$$

where

$$M_{jn}{}^m = \int_{\cos\alpha}^{1} x \, P_{k_j}{}^m(x)P_{k_n}{}^m(x) \, dx \tag{7}$$

$$G_{s,t}{}^m = \int_{\cos\alpha}^{1} P_s{}^m(x) \, P_t{}^m(x) \, dx \tag{8}$$

and $G^m_{k_j,k_n} = 0$ if $j \neq n$ (see Ref. 2). It can be shown[5] that for any integers, $n \geq m \geq 0$, $i \geq m \geq 0$, one has

$$(2i + 1)E_{i,m}{}^m(x) = (2m - 1)(i + m - 1)(i + m)E^{m-1}_{i-1,m-1}(x) - (2m - 1)(i - m + 2)(i - m + 1)E^{m-1}_{i+1,m-1}(x) \tag{9}$$

$$(n - m)(2i + 1)E_{i,n}{}^m(x) = (2n - 1)(i - m + 1)E^m_{i+1,n-1}(x) + (2n - 1)(i + m)E^m_{i-1,n-1}(x) - (2i + 1)(n + m - 1)E^m_{i,n-2}(x) \tag{10}$$

where

$$E_{i,n}{}^m(x) = \int P_i{}^m(x) \, P_n{}^m(x) \, dx \tag{11}$$

and $E_{i,n}{}^m = 0$ if $n < m$ or $i < m$. Therefore,

$$G_{s,t}{}^m = E_{s,t}{}^m(1) - E_{s,t}{}^m(\cos\alpha)$$

For a given m, $E_{i,m}{}^m(x)$ can be constructed readily by Eq. (9), using the starting polynomials $E_{i,0}{}^0(x) = \int P_i(x) \, dx$, and then one can proceed to build $E_{i,n}{}^m(x)$ by Eq. (10).

For $L_{jj}{}^m$, it is seen at once that $L_{jj}{}^m = G^m_{k_j,k_j}$. The cases of hemispheres and spheres correspond to $\alpha = \pi/2$ and $\alpha = \pi$, respectively.

References

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Water Impact of the Mercury Capsule: Correlation of Analysis with NASA Tests

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REFERENCE 1 is essentially an extension of the von Kármán approach for rotationally constrained prismatic bodies impacting smooth water. The method presented accounts for the behavior of a pitching nonprismatic vehicle penetrating rough water. The mathematical model is roughly similar to that for seaplane hulls.

In the absence of a more suitable method, the procedure proposed in Ref. 1 has been applied to the water landings of the Apollo command module. To establish the applicability of the technique to such a configuration, a comparison of predicted results for the Mercury capsule with experimental values from NASA[2] has been made. The results of this correlation are presented herein.

The mathematical model employed in the procedure is that shown in Fig. 1. The spherical bottom of the Mercury capsule is represented by a series of wedges, each with a 10° deadrise angle. Because of the constant deadrise, the chine heights

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